Algebraic and arithmetic area for m planar Brownian paths

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Abstract

The leading and next to leading terms of the average arithmetic area $\langle S(m) \rangle$ enclosed by $m \to \infty$ independent closed Brownian planar paths, with a given length t and starting from and ending at the same point, is calculated. The leading term is found to be $\langle S(m) \rangle \sim \frac{\pi t}{2} \ln m$ and the 0-winding sector arithmetic area inside the m paths is subleading in the asymptotic regime. A closed form expression for the algebraic area distribution is also obtained and discussed.

1   Introduction

The question of the area, be it algebraic A or arithmetic S, enclosed by a closed Brownian planar path, of a given length t and starting from and ending at a given point, is an already old subject. It started in the mid-twentieth century with the well-known Levy's law [1] for the probability distribution $P(A)$ of the algebraic area. The question of the probability distribution $P(S)$ of the arithmetic area is a much more difficult issue pertaining to its non local nature. Interestingly enough, the complete calculation of the first moment of the distribution, $\langle S \rangle = \pi t/5$, has been only recently achieved by SLE technics [2]. Since on the other hand the average arithmetic area $\langle S_n \rangle = t/(2\pi n^2)$ of the $n \neq 0$-winding sectors inside the curve has been known for some time [3] thanks to path integral technics, one can readily deduce from these results that the average arithmetic area $\langle S_0 \rangle$ of the 0-winding sectors[4] inside the curve is $\pi t/30$. A n-winding sector is defined as a set of points enclosed n times by the path and a 0-winding sector is made of points which are either outside the path, or inside it but enclosed an equal number of times clockwise and anti-clockwise. Note finally that, having in mind more simple winding properties, the asymptotic probability distribution at large time of the angle spanned by one path around a given point is also known [5].

Clearly, the random variables S_n and S_0 are such that $S = \sum_{n=-\infty}^{\infty} S_n$ and $A = \sum_{n=-\infty}^{\infty} n S_n$. They happen to be the basic objects needed to define quantum mechanical models where random magnetic impurities are modelised by Aharonov-Bohm vortices [6]. Recently some progresses have been made on the geometrical structure of the n-winding sectors thanks to numerical simulations of the Hausdorff dimension of their fractal perimeter [7].

The question addressed in the present work concerns the generalisation of $\langle S \rangle = \pi t/5$ to the arithmetic area $\langle S(m) \rangle$ spanned by m independent closed paths of a given length t starting and returning at the same point. More precisely, one would like to have some information on the scaling of $\langle S(m) \rangle$ when $m \to \infty$. Using again path integral technics in the line of [3], one will show that it is possible to compute exactly the leading and next to leading asymptotic terms of $\langle S(m) - S_0(m) \rangle$ where the contribution of the 0-windings sectors inside the m-paths has been substracted precisely for the reason discussed above in the one path case. One will find that the leading asymptotic term scales like $\ln m$, namely $\langle S(m) - S_0(m) \rangle \sim \frac{\pi t}{2} \ln m$ with, as already said, no information so far on $\langle S_0(m) \rangle$ inside the paths.

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[4]The path integral approach diverges for $n = 0$ because it cannot distinguish zero-winding sectors inside the path from the outside of the path, the latter being of infinite area and, trivially, also a 0-winding sector.

To go a little bit further, one might consider a simplification of the problem by looking at the arithmetic area of the convex envelop of the paths, a simpler geometrical object than their actual fractal envelop. It is known [8] that the area of the convex envelop of one path is $\langle S \rangle_{\text{convex}} = \pi t/2$. Recently, the asymptotic behavior for m paths when $m \to \infty$ has been found [9] to be precisely $\langle S(m) \rangle_{\text{convex}} \sim \frac{\pi t}{2} \ln m$. In light of the identical scaling obtained for $\langle S(m) - S_0(m) \rangle$, it means that, in geometrical terms, the m paths tend to fully occupy the area available inside their convex envelop when m is large. Clearly, $\langle S(m) \rangle_{\text{convex}}$ being by construction bigger than the actual $\langle S(m) \rangle$, the latter lies between $\langle S(m) - S_0(m) \rangle$ and $\langle S(m) \rangle_{\text{convex}}$. Now, since the leading asymptotic behavior of both the lower and upper bounds are found to be equal, in the asymptotic regime one concludes necessarily that $\langle S(m) \rangle \sim \frac{\pi t}{2} \ln m$ and that $\langle S_0(m) \rangle$ is subleading[5].

2   Algebraic area distribution

As a warm up, let us consider the generalization of Levy's law to m independent paths. Let us first start with one path of length t, starting from and ending to a given point $\vec{r}$, so that $\vec{r}(0) = \vec{r}(t) = \vec{r}$. The algebraic area enclosed by this path is $A = \vec{k} . \int_0^t \vec{r}(\tau) \wedge d\vec{r}(\tau)/2$ where $\vec{k}$ is the unit vector perpendicular to the plane. The path integral leads to

$$\langle e^{iBA} \rangle = \frac{\mathcal{G}_B(\vec{r}, \vec{r})}{\mathcal{G}_0(\vec{r}, \vec{r})} \tag{1}$$

where $\mathcal{G}_0(\vec{r}, \vec{r}) = 1/(2\pi t)$ and

$$\mathcal{G}_B(\vec{r}, \vec{r}) \equiv \int_{\vec{r}(0)=\vec{r}}^{\vec{r}(t)=\vec{r}} \mathcal{D}\vec{r}(\tau) e^{-\frac{1}{2} \int_0^t \dot{\vec{r}}^2(\tau) \mathrm{d}\tau + iB\vec{k} . \int_0^t \frac{\vec{r}(\tau) \wedge d\vec{r}(\tau)}{2}} = \frac{1}{2\pi t} \frac{\frac{Bt}{2}}{\sinh\left(\frac{Bt}{2}\right)} \tag{2}$$

In (1) the average $\langle \ \ \rangle$ has been made over the set C of all paths of length t starting from and ending at $\vec{r}$. As a result $\mathcal{G}_B$ is the Landau propagator of a charged particle in an uniform magnetic field $\vec{B} = B\vec{k}$. Fourier transforming, the probability distribution of A -the Levy's law- is

$$P(A) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\frac{Bt}{2}}{\sinh\left(\frac{Bt}{2}\right)} e^{-iBA} \mathrm{d}B = \frac{\pi}{2t} \frac{1}{\cosh^2\left(\frac{\pi A}{t}\right)} \tag{3}$$

In the case of m paths, one should compute instead of (3)

$$P_m(A) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\frac{Bt}{2}}{\sinh\left(\frac{Bt}{2}\right)}\right)^m e^{-iBA} \mathrm{d}B \tag{4}$$

The integral in (4) can be performed by rewriting $1/\sinh(Bt/2)^m = 2^m e^{-mBt/2}/(1 - e^{-Bt})^m$ and by expanding, when the integration variable $B > 0$, the denominator in powers of e^{-Bt}. One obtains finally

$$P_m(A) = \frac{m!}{2\pi t} \sum_{k=0}^{\infty} \binom{k + m - 1}{k} \left(\frac{1}{(k + m/2 + iA/t)^{m+1}} + \text{cc}\right) \tag{5}$$

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[5]This can be easily understood by noticing that 0-winding sectors tend to disappear when more and more paths overlap as m increases, which also implies that n-winding indices tend to increase.
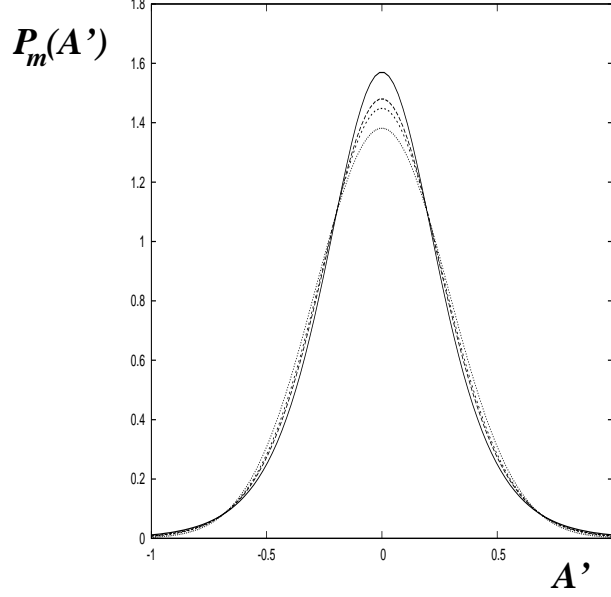
Figure 1: The distribution $P_m(A')$ for the rescaled algebraic area $A' = A/(t\sqrt{m})$. Starting from the curve with the greatest maximum $m = 1$ (Levy's law eq.(3)), $m = 2$ (eq. (7)), $m = 3$ (eq. (8)), $m = \infty$ (eq. (6)).

where the complex conjugate term corresponds to the $B < 0$ integration (it amounts to set $A \to -A$ in the $B > 0$ integration result).

Noticing that A is the sum of m independent random variables each satisfying Levy's law, one expects A to be gaussian when m becomes large. Indeed, one can observe that, when $m \to \infty$, the main contribution to the integral in (4) comes from small B values. Thus,

$$\left(\frac{\frac{Bt}{2}}{\sinh\left(\frac{Bt}{2}\right)} \right)^m \sim \left(\frac{1}{1 + \frac{B^2 t^2}{24}} \right)^m \sim e^{-m \frac{B^2 t^2}{24}}.$$

Rescaling the area as $A' = A/(t\sqrt{m})$, (4) leads to

$$P_\infty(A') = \sqrt{\frac{6}{\pi}} e^{-6A'^2} \tag{6}$$

In terms of A', we get from (5) or from contour integration,

$$P_2(A') = \frac{\pi\sqrt{2}}{\sinh^2(\pi\sqrt{2}A')} \left(\pi\sqrt{2}A' \coth(\pi\sqrt{2}A') - 1 \right) \tag{7}$$

$$P_3(A') = \frac{\pi\sqrt{3}}{2\cosh^2(\pi\sqrt{3}A')} \left(3 - 6\pi\sqrt{3}A' \tanh(\pi\sqrt{3}A') - ((\pi\sqrt{3}A')^2 + \frac{\pi^2}{4})(1 - 3\tanh^2(\pi\sqrt{3}A')) \right) \tag{8}$$

Figure 1 clearly shows that $P_m(A')$ converges quickly to the gaussian $P_\infty(A')$ when m becomes large. In particular A scales like $\sqrt{m}$ when $m \to \infty$. The scaling will be different for the arithmetic area discussed in the next section.

3

3 Winding properties and arithmetic area for m Brownian paths

3.1 The case of one path: notations and known results.

The arithmetic area enclosed by a planar Brownian path is closely related to its winding properties. Let us again consider a path of length t, starting from and ending at $\vec{r}$, and let us set θ to be the angle wounded by the path around a fixed point, say the origin O. Again the average $\langle e^{i\alpha\theta} \rangle$ is made over C

$$\langle e^{i\alpha\theta} \rangle = \frac{\mathcal{G}_\alpha(\vec{r}, \vec{r})}{\mathcal{G}_0(\vec{r}, \vec{r})} \tag{9}$$

where

$$\mathcal{G}_\alpha(\vec{r}, \vec{r}) \equiv \int_{\vec{r}(0)=\vec{r}}^{\vec{r}(t)=\vec{r}} \mathcal{D}\vec{r}(\tau) e^{-\frac{1}{2}\int_0^t \dot{\vec{r}}^2(\tau)\mathrm{d}\tau + i\alpha\int_0^t \dot{\theta}(\tau)\mathrm{d}\tau} = \frac{1}{2\pi t}e^{-\frac{r^2}{t}}\sum_{k=-\infty}^{+\infty} I_{|k-\alpha|}\left(\frac{r^2}{t}\right) \tag{10}$$

By symmetry the average depends only on r with $\mathcal{G}_\alpha$ being the propagator of a charged particle coupled to a vortex at the origin (the $I_{|k-\alpha|}$'s are modified Bessel functions). Obvious symmetry and periodicity considerations such as $\mathcal{G}_\alpha = \mathcal{G}_{\alpha+1} = \mathcal{G}_{1-\alpha}$ allow to restrict to $0 \le \alpha \le 1$. We also set $r^2/t \equiv x$ (so that $2\pi r\mathrm{d}r = \pi t\mathrm{d}x$) and $\langle e^{i\alpha\theta} \rangle \equiv G_\alpha(x)$ with

$$G_\alpha(x) = e^{-x}\sum_{k=-\infty}^{+\infty} I_{|k-\alpha|}(x) \tag{11}$$

so that $G_0(x) = 1$.

To integrate over $\vec{r}$ -while keeping the position of the vortex fixed at the origin- is the same as to integrate over the vortex position -while keeping the starting and ending point of the path fixed. Therefore it amounts to count the arithmetic areas S_n of the n-winding sectors, the 0-winding sector included

$$\int_0^\infty \pi t\mathrm{d}x\, G_\alpha(x) = \sum_{n\neq 0}\langle S_n\rangle e^{i\alpha 2\pi n} + \langle \tilde{S}_0\rangle \tag{12}$$

This integral diverges since, as already stressed in the introduction, $\tilde{S}_0$ is the sum of S_0, the arithmetic area of the 0-winding sectors enclosed by the path, and of the area of the outside of the path, which is infinite. However, since formally for $\alpha = 0$,

$$\int_0^\infty \pi t\mathrm{d}x\, G_0(x) = \sum_{n\neq 0}\langle S_n\rangle + \langle \tilde{S}_0\rangle \tag{13}$$

then

$$Z_\alpha \equiv \pi t\int_0^\infty \mathrm{d}x\,(1 - G_\alpha(x)) = \sum_{n\neq 0}\langle S_n\rangle\left(1 - e^{i\alpha 2\pi n}\right) \tag{14}$$

is finite. One deduces

$$\langle S_n\rangle = -\int_0^1 Z_\alpha e^{-i\alpha 2\pi n}\mathrm{d}\alpha \qquad n \neq 0 \tag{15}$$

and

$$\langle S - S_0\rangle \equiv \sum_{n\neq 0}\langle S_n\rangle = \int_0^1 Z_\alpha\mathrm{d}\alpha \tag{16}$$

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Using the Laplace transform of the modified Bessel functions, we readily recover

$$Z_\alpha = \pi t \alpha (1 - \alpha) \tag{17}$$

$$\langle S_n \rangle = \frac{t}{2\pi n^2} \qquad n \neq 0 \tag{18}$$

$$\langle S - S_0 \rangle = \frac{\pi t}{6} \tag{19}$$

3.2   The case of m independent paths.

It is easy to realize that $\mathcal{G}_\alpha(\vec{r}, \vec{r})^m$ provides the appropriate measure for counting the sets of m closed paths of length t starting from and ending at $\vec{r}$. Following the same line of reasoning as in section 3.1, we get

$$Z_\alpha(m) \equiv \pi t \int_0^\infty \mathrm{d}x \, (1 - G_\alpha(x)^m) = \sum_{n \neq 0} \langle S_n(m) \rangle \left(1 - e^{i\alpha 2\pi n}\right) \tag{20}$$

and

$$\langle S(m) - S_0(m) \rangle \equiv \sum_{n \neq 0} \langle S_n(m) \rangle = \int_0^1 Z_\alpha(m) \mathrm{d}\alpha \tag{21}$$

where $S(m)$, $S_n(m)$ and $S_0(m)$ stand respectively for the total, n-winding sectors and 0-winding sectors arithmetic areas enclosed by the m paths.   For example a sector of points which have been enclosed once by one path in the clockwise direction, twice by another path in the anticlockwise direction, and not enclosed by the other $m - 2$ paths, has winding number $n = -1 + 2 = 1$.

To find the leading behavior of $S(m)$ in the large m limit, one has to evaluate $G_\alpha(x)^m$ when $m \to \infty$, and so one needs a tractable expression for $G_\alpha(x)$. Starting from (11) and using again the Laplace transform of Bessel functions, we get

$$\frac{\mathrm{d}G_\alpha(x)}{\mathrm{d}x} = \frac{2}{\sqrt{\pi}} \sin(\pi\alpha) e^{-2x} (2x)^{\alpha-1} U\left(\alpha - \frac{1}{2}; 2\alpha; 2x\right) \tag{22}$$

where U is the degenerate hypergeometric function [10]

$$U(a; b; z) = \frac{z^{-a}}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} \left(1 + \frac{t}{z}\right)^{b-a-1} \mathrm{d}t \tag{23}$$

It is easy to verify that $G_\alpha(x) = G_{1-\alpha}(x)$, as it should.

From (22)

$$G_\alpha(\infty) = G_\alpha(0) + \int_0^\infty \frac{\mathrm{d}G_\alpha(x)}{\mathrm{d}x} \mathrm{d}x = 1 \tag{24}$$

as it should far away from the origin (one has used that when $\alpha \neq 0$, $G_\alpha(0) = 0$).   More precisely, since for z large $U(a; b; z) \sim z^{-a}$, one has

$$1 - G_\alpha(x) \sim \frac{\sin(\pi\alpha)}{\sqrt{2\pi x}} e^{-2x} \qquad \text{when} \qquad x \to \infty \tag{25}$$

On the other hand, for $0 < \alpha < 1/2$,

$$G_\alpha(x) \sim \frac{\left(\frac{x}{2}\right)^\alpha}{\Gamma(\alpha + 1)} \qquad \text{when} \qquad x \to 0 \tag{26}$$

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Changing variable $2x = y \ln m$ in (20)

$$Z_\alpha(m) = \frac{\pi t}{2} \ln m \int_0^\infty dy \left(1 - \left(G_\alpha(\frac{y \ln m}{2}) \right)^m \right) \tag{27}$$

one sees that in the limit $m \to \infty$

- when $y = 0$, that is $y \ln m = 0$, $(G_\alpha)^m = 0$

- when $y > 0$, that is $y \ln m \to \infty$, the asymptotics (25) can be used so that

$$\left(G_\alpha(\frac{y \ln m}{2}) \right)^m \to \left(1 - \frac{\sin(\pi\alpha)}{\sqrt{\pi y \ln m}} m^{-y} \right)^m \sim e^{-\frac{\sin(\pi\alpha)}{\sqrt{\pi y \ln m}} m^{1-y}} \tag{28}$$

It follows that

$$\left(G_\alpha(\frac{y \ln m}{2}) \right)^m \to_{m \to \infty} \Theta(y - 1) \tag{29}$$

where Θ is the Heaviside function. Finally

$$Z_\alpha(m) \sim \frac{\pi t}{2} \ln m \tag{30}$$

and

$$\langle S(m) - S_0(m) \rangle \sim \frac{\pi t}{2} \ln m \tag{31}$$

As already discussed in the Introduction, it means that since $\langle S(m) \rangle_{\text{convex}} \sim \frac{\pi t}{2} \ln m$, at leading order

$$\langle S(m) \rangle \sim \frac{\pi t}{2} \ln m \tag{32}$$

and, necessarily, $\langle S_0(m) \rangle$ is subleading.

In Figure 2, numerical simulations (random walks on a square lattice) for $\langle S(m) \rangle_{\text{convex}}/t$, $\langle S(m) \rangle/t$ and $\langle S(m) - S_0(m) \rangle/t$ are displayed. One sees that if $\langle S(m) \rangle_{\text{convex}}/t$ converges rapidly to $\frac{\pi}{2} \ln m$, this is not the case for $\langle S(m) \rangle/t$ and $\langle S(m) - S_0(m) \rangle/t$. It means that subleading corrections are needed. They originate from the fact that, when m is large but not infinite, $(G_\alpha)^m$ deviates from a Heaviside function. One should compute

$$\frac{1}{t} \langle S(m) - S_0(m) \rangle = \frac{\pi}{2} \ln m + \frac{\pi}{2} \ln m \, (c_1 + c_2) \tag{33}$$

with

$$c_1 = \int_0^1 d\alpha \int_1^\infty dy \left(1 - \left(G_\alpha(\frac{y \ln m}{2}) \right)^m \right) \tag{34}$$

$$c_2 = - \int_0^1 d\alpha \int_0^1 dy \left(G_\alpha(\frac{y \ln m}{2}) \right)^m \tag{35}$$

For $y > 1$,

$$1 - \left(G_\alpha(\frac{y \ln m}{2}) \right)^m \sim m \frac{\sin(\pi\alpha)}{\sqrt{\pi y \ln m}} m^{-y} \tag{36}$$

so that

$$\frac{\pi}{2} \ln m \, c_1 \sim \frac{1}{\sqrt{\pi \ln m}} \tag{37}$$

For $0 < y \le 1$,

$$\left(G_\alpha(\frac{y \ln m}{2}) \right)^m \sim e^{-am^{1-y}} \tag{38}$$
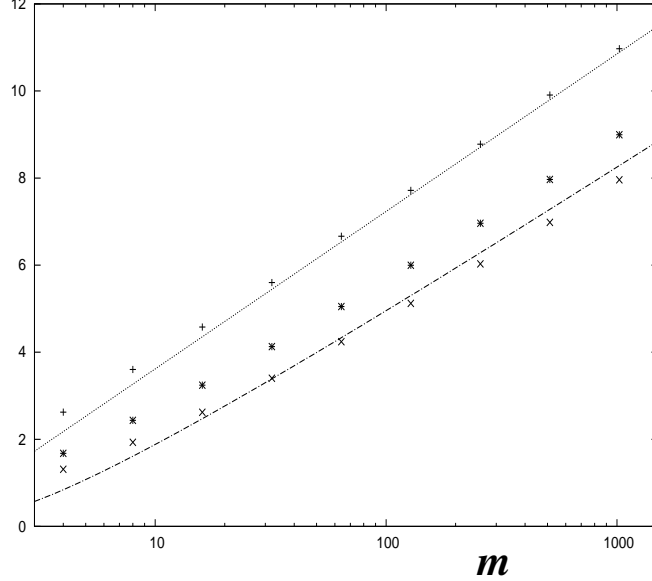
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Figure 2: The average arithmetic area: i) numerical simulations (10000 events) of closed random walks of 10^6 steps on a square lattice; $m = 4, 8, 16, ..., 1024$; +: $\langle S(m)\rangle_{\text{convex}}/t$, *: $\langle S(m)\rangle/t$, ×: $\langle S(m) - S_0(m)\rangle/t$ ii) analytical results for $\langle S(m) - S_0(m)\rangle/t$; upper continuous line: $\pi \ln(m)/2$ (leading order); lower dotted line: eq. (40) that includes the subleading corrections.

with $a = \sin(\pi\alpha)/\sqrt{\pi y \ln m} \approx \sin(\pi\alpha)/\sqrt{\pi \ln m}$ since when m is large, y is peaked to 1. Changing variable to $a\, m^{1-y} = z$ and integrating over z, then over α, leads to

$$\frac{\pi}{2} \ln m\, c_2 \sim -\frac{\pi}{4} \ln\ln m - \frac{\pi}{2}\left(\ln\sqrt{4\pi} - C\right) - \frac{1}{\sqrt{\pi \ln(m)}} + \text{subleading} \tag{39}$$

where C is the Euler constant. Collecting all terms, we finally obtain

$$\frac{1}{t}\langle S(m) - S_0(m)\rangle = \frac{\pi}{2} \ln m - \frac{\pi}{4} \ln\ln m - \frac{\pi}{2}\left(\ln\sqrt{4\pi} - C\right) + \text{subleading} \tag{40}$$

One sees in Figure 2 that the subleading corrections greatly improve the fit: the lower dotted line (40) is indeed not far from the numerical data (the agreement is of course not entirely perfect but further subleading corrections seem hard to reach).

4   Conclusion

In conclusion one has established that the leading behavior of the average arithmetic area $\langle S(m)\rangle$ enclosed by $m \to \infty$ independent closed Brownian planar paths is $\frac{\pi t}{2} \ln m$. The algebraic area, on the other hand, scales like $t\sqrt{m}$. One should stress that the quite different asymptotic behaviors pertain to the essentially different nature of the areas considered: the algebraic area is additive, which is not the case of the arithmetic area. On another front, it remains a real challenge to get some information on the subleading asymptotic behavior of the 0-winding sector area $\langle S_0(m)\rangle$. Again path integral technics are not adapted to this case, whereas SLE machinery should in principle work.

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